Construction of a General Spin Foam Model of Lorentzian BF theory and Gravity

Suresh K Maran

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Abstract

In this article we report our progress in the construction of a general Lorentzian spin foam model based on the Gelfand-Naĭmark theory of the representations of $SL(2,\boldsymbol{C})$ which might include both previously proposed the $SL(2,\boldsymbol{C})/SU(2)$ based Barrett-Crane model and the $SL(2,\boldsymbol{C})/SU(1,1).Z_2$ based Rovelli-Perez model. First we construct the simplex amplitude for the BF $SL(2,\boldsymbol{C})$ model. Then we discuss the asymptotic limit of this model. Next we discuss the implemention of the Barrett-Crane constraints on this model. We derive an equation that the general Lorentzian spin foam model has to satisfy. In the appendix we give a simple derivation of the Clebsch-Gordan coefficients for $SL(2,\boldsymbol{C})$.

1 Introduction

A candidate for the spin foam model[1] of Lorentzian gravity was proposed by Barrett and Crane [3]. This model was constructed based on harmonic analysis on the homogenous space $H^+ = SL(2,\boldsymbol{C})/SU(2)$ which is the upper sheet of the two-sheet hyperboloid in four dimensional Minkowski space-time. Later Rovelli and Perez proposed a way of deriving this model using a field theory over group formulation [4]. But, Perez and Rovelli [5] also proposed an alternative spin foam model based on the homogenous space $SL(2,\boldsymbol{C})/U^{(-)}$ where $U^{(-)} = SU(1,1) \otimes Z_2$. This suggests that there must be a model which contains both these models. We call this model the general model while we call the Rovelli-Perez and the Barrett-Crane model the partial models.

To construct a Lorentzian spin foam model both Barrett-Crane and Rovelli-Perez proceed by analogy with the construction of the Riemmanian spin foam model. To explain our method for construction of the general model, let us discuss first the construction of the spin foam model of Riemannian gravity model. The first step is to do the path integral quantization of the $SO(4)$ BF theory [6] using its discrete action to get its spin foam model [**?**]. Then the next step is to impose the Barrett-Crane constraints at the quantum level. This is supposed to

[1]We refer to [1] for a nice introduction to spin foam models. We also refer to [2] for an up-to-date review of spin foam models and for references.

result in the spin foam model of the Riemannian gravity [2], [8], [9], [10], [11],. The third step is to rewrite the simplex amplitude in terms of certain propagators in the homogenous space S^3. To construct a Lorentzian spin foam model both Barrett-Crane and Rovelli-Perez proceed by analogy with the third step. But the problem with this approach is, that there are three possible subgroups for $SL(2,\boldsymbol{C})$, which could be used for a Barrett-Crane type construction namely $SU(2)$, $E(2)$ and $SU(1,1)$. Because of this, the analogy cannot suggest a unique Lorentzian spin foam model. We here construct the Lorentzian spin foam model starting from the first step using the Gelfand-Naĭmark representation theory of $SL(2,\boldsymbol{C})$ [12]. The assumption is that this method would yield a general spin foam model that contain the spin foams relating to all the subgroups.We believe it is important to do this for two reasons. First, it is good to relate a model to an action Second, its good to have all the three subgroups $SU(2)$, $E(2)$ and $U(1,1)$ of $SL(2,\boldsymbol{C})$ play a role. Because these groups are physically related to particles which are massive, massless and tachyonic. Even though the third one is not physical, we donot know what role this type of contribution would play in quantum gravity in future. If the tachyonic part of our theory is not physical, we want our theory itself predict that, instead of our artificially excluding it. At this point it is also not clear what role the $E(2)$ will play in the theory. We believe the study of the asymptotic limit [13], [14], the continuum limit and the inclusion of matter will shed light on this. Here we restrict ourselves to the construction of spin foam models only.

In section two we discuss how to construct the spin foam model of the $SL(2,\boldsymbol{C})$ BF theory using the Gelfand- Naĭmark representation theory of $SL(2,\boldsymbol{C})$ and analyze the asymptotic limit of the BF spin foam model. We formally derive the equations that describe the asymptotic limit of the theory. In section three we discuss how to impose the Barrett-Crane constraints on the BF spin foam model to get a general Lorentzian spin foam model of gravity. We present an equation that needs to be solved to get the general Lorentzian model. In section four we present two issues that need to be addressed in the construction of Lorentzian spin foam models. We have tried to make this article as much self contained as possible.

2 The Spin Foam for the SL(2,C) BF Theory

Our presentation here in the first two subsections follows that of Baez [1], [9]. The new ingredient is the use of the Gelfand-Naĭmark represention theory of $SL(2,C)$. Advanced readers might be able to glance through the first subsections.and the earlier part of the second subsection.

2.1 From the discrete action

The Spin foam model for the $SL(2,\boldsymbol{C})$ BF theory action can be derived from the discretized BF action by using the path integral quantization [**?**], [1]. Let M be a simplicial manifold. Let $g_e \in SL(2,\boldsymbol{C})$ be the discretized connection

associated to the edges (three-simplices) and $H_b = \prod_{e \supset b} g_e$ be the holonomy around a bone (two-simplices). Then the discrete BF action is

$$S_d = \sum_b tr(B_b H_b).$$

Here $B_b \in sl(2, \boldsymbol{C})^*$ is the discrete analog of B. The B_b are 4×4 antisymmetric complex matrices. Then the quantum partition function is calculated using the path integral formulation as [2]

$$\begin{aligned} Z &= \int \prod_b dB_b \exp(i \operatorname{Re}(S_d)) \prod_e dg_e \\ &= \int \prod_b \delta(H_b) \prod_e dg_e, \end{aligned} \tag{2.1}$$

where dg_e is the invariant measure on the group $SL(2, \boldsymbol{C})$. A summary of the Gelfand-Naĭmark representation theory of $SL(2, \boldsymbol{C})$ and relevant references are given in appendix A.

Now consider the identity [4]

$$\delta(g) = \frac{1}{8\pi^4} \int \chi \bar{\chi} tr(T_\chi(g)) d\chi, \tag{2.2}$$

where the $T_\chi(g)$ is the $\chi = n + i\rho$ unitary representation of $SL(2, \boldsymbol{C})$. The Integration with respect to $d\chi$ in the above equation is interpreted as the summation over an integer n and the integration over a real variable ρ. The $|\chi| = \sqrt{n^2 + \rho^2}$ is the analog of the dimension of a representation of a compact group.

Substituting this into equation (2.1) we get

$$\begin{aligned} Z &= \int \prod_b \left[\int \frac{\chi_b \bar{\chi}_b}{8\pi^4} tr(T_{\chi_b}(\prod_{e \supset b} g_e)) d\chi_b \right] \prod_e dg_e \\ &= \int \left[\left(\prod_b \frac{\chi_b \bar{\chi}_b}{8\pi^4} \right) \left(\int \prod_b dg_e tr(\prod_{e \supset b} T_{\chi_b}(g_e)) \right) \right] \prod_b d\chi_b \\ &= \int \left[\left(\prod_b \frac{\chi_b \bar{\chi}_b}{8\pi^4} \right) Tr \left(\prod_e \int dg_e \bigotimes_{b \subset e} T_{\chi_b}(g_e) \right) \right] \prod_b d\chi_b, \end{aligned} \tag{2.3}$$

where Tr denotes the required trace operations from the trace operations in the previous line. The integrand of the quantity in the second parentheses is the g_e integration of the tensor product of the representation matrices $\rho_{\chi_b}(g_e)$ that

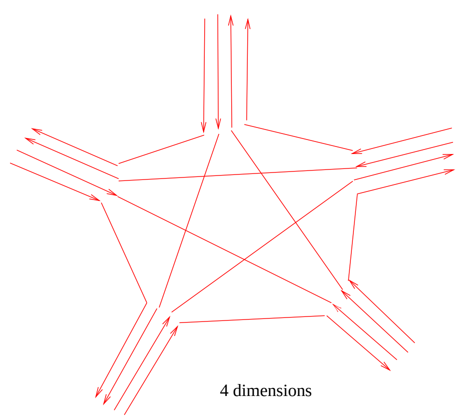
[2] While calculating the path integral only the real part of the action is being used. Otherwise the integration with respect to the B_b variables no longer leads to the condition that the curvature (holonomy) is zero (identity) as required by the equations of motion of the BF theory, if the group elements are complex. The physical consequences of this procedure have to be investigated.

are part of the holonomy around the four bones of the edge e. This quantity can be rewritten as a product of the intertwiners i_e for $SL(2, \boldsymbol{C})$ as follows

$$\int dg \bigotimes_{b \supset e} T_{\chi_b}(g) = \sum_{i_e} i_e \bar{\imath}_e. \tag{2.4}$$

The integral on the left hand side of this equation will be referred to as the *edge integral*. The bar denotes the complex conjugation. In the right hand side of the edge integral there is a sum of product of two intertwiners. Each of the intertwiners corresponds to one of the sides of the edge.

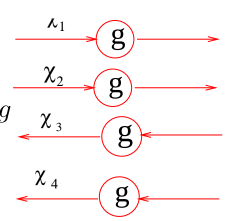
Above, it is assumed that the holonomies pass through the edge in the same direction. But usually their directions can be random. Reversing the direction of a holonomy is equivalent to complex conjugating (the inverse of the transpose) the representations in the edge integral. To simplify the calculation of the edge integrals, the directions of the holonomies can be chosen as illustrated below for a simplex. The parallel sets of arrows indicate the direction in which the holonomies are traversed through the edges of a simplex.



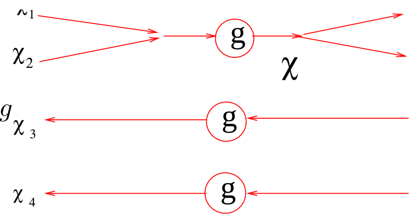
The corresponding edge integral given below is calculated as follows,

$$I_4 = \int dg T_\chi(g)(x_1, x_2) T_\lambda(g)(y_1, y_2) \bar{T}_\mu(g)(z_1, z_2) \bar{T}_\mu(g)(z_1, z_2)$$

$$= \int dg$$



$$= \int d^2\chi \int dg$$



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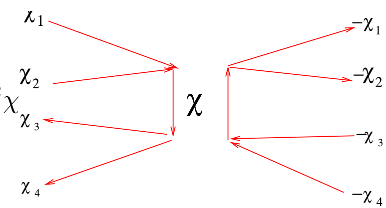
From appendix B we have

$$I_4 = \int d^2\chi$$



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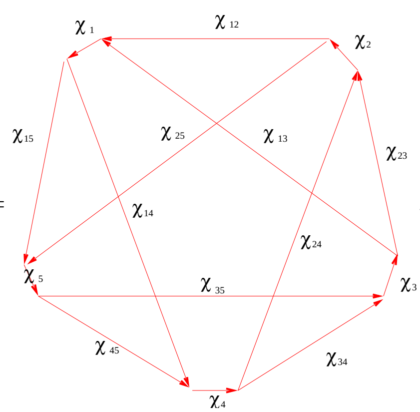
The nodes where the three links meet are the Clebsch-Gordan coefficients of $SL(2, C)$. In the next section we discuss these coefficients. When this edge integral formula is used in equation (2.3) and all the required trace operations are performed, it is seen that each open link of each intertwiner corresponding to an inner side of an edge of each simplex, only integrates with the link of an intertwiner corresponding to an inner side of another edge of the same simplex. The partition function Z splits into a product of terms, with each term interpreted as a quantum amplitude associated to a simplex in the triangulation.

The quantum amplitude associated to each simplex s is given below and will be referred to as the $\{15\chi\}$ symbol,

$$\{15\chi\} =$$



$.$

The $\frac{\chi_b \bar{\chi}_b}{8\pi^4}$ term is the quantum amplitude associated to the bone b. The final partition function is

$$Z = \int \prod_b \frac{\chi_b \bar{\chi}_b}{8\pi^4} \prod_s Z(s) \prod_b d\chi_b, \tag{2.5}$$

where $Z(s)$ is interpreted as the amplitude for an n-simplex s and d_{χ_b} is interpreted as the amplitude of the bone b, $D\chi = \prod_b d\chi_b \prod_e d\chi_e$. The integration with the measure $d\chi$ ($\chi = m + i\rho$) is understood as a summation of over m and ordinary integration with respect to ρ. Here χ_e is the internal representation used to define the intertwiner $d\chi_e$. This partition function may not be finite in general.

2.2 Using the Quantization of the Edges.

We believe, this version of the derivation was originally introduced for the Riemannian Gravity by Barbieri [15], and further developed by Baez, Barrett and Crane [9], [16], [8]. These authors mainly focussed on the group $SO(4)$. Here we apply this method that for the group $SO(3,1) \approx SL(2,\boldsymbol{C})$.

Let $B_i \in sl(2,\boldsymbol{C})^*$ where $i = 1,2,3,4$ be the discrete variables associated to the four bones (triangles) of an edge (tetrahedron) of a four-simplex. Consider the equation:

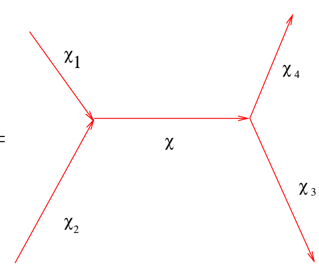
$$B_1 + B_2 + B_3 + B_4 = 0.$$

This is the closure constraint. This is the discrete version of the equation $dB = 0$ which is one of the field equations of the continuum BF theory. This equation can be quantized as follows. To each bone i of the edge associate a Hilbert space H_i (which is the linear space that carries the unitary representation of $SL(2,\boldsymbol{C})$) on which the quantum generators $\hat{B}_i$ (which are simply the unitary representations of generators of $SL(2,\boldsymbol{C})$) act. Then the previous equation can be promoted to a quantum constraint,

$$\left(\hat{B}_1 + \hat{B}_2 + \hat{B}_3 + \hat{B}_4\right)\psi = 0,$$

where $\psi \in H_1 \otimes H_2 \otimes H_3 \otimes H_4$. The solutions of this equation are the intertwiners i_e,

$\psi =$



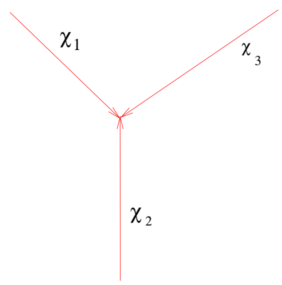
, (2.6)

where it is assumed that each edge i is associated a D_{χ_i} the linear space of unitary representation χ_i.

Each solution of the constraint is a quantum state that depends firstly on the five complex numbers $\chi_1, \chi_2, \chi_3, \chi_4$ and χ each of which labels a unitary representation of $SL(2,\boldsymbol{C})$ and secondly on a 2-2 split of the four bones in the edges. In the above diagram a 12-34 split is being used.

In the graph, in equation, the three arrowed links intersecting at the nodes are the Clebsch-Gordan coefficients of $SL(2,\boldsymbol{C})$. Changing the direction of the arrows is equivalent to complex conjugating the associated representation.

The Clebsch-Gordan coefficients were originally derived by Naĭmark [17]. A quick way to calculate them is shown in appendix B. The Clebsch-Gordan coefficients are explicitly



$$= C(\chi_1, \chi_2, \chi_3, z_1, z_2, z_3)$$

$$= (z_1 - z_2)^{\frac{-\chi_1-\chi_2+\chi_3}{2}-\frac{1}{2}} (z_2 - z_3)^{\frac{\chi_1-\chi_2-\chi_3}{2}-\frac{1}{2}} (z_3 - z_1)^{\frac{-\chi_1+\chi_2-\chi_3}{2}-\frac{1}{2}}$$
$$(\bar{z}_1 - \bar{z}_2)^{-\frac{-\bar{\chi}_1-\bar{\chi}_2+\bar{\chi}_3}{2}-\frac{1}{2}} (\bar{z}_2 - \bar{z}_3)^{-\frac{\bar{\chi}_1-\bar{\chi}_2-\bar{\chi}_3}{2}-\frac{1}{2}} (\bar{z}_3 - \bar{z}_1)^{-\frac{-\bar{\chi}_1+\bar{\chi}_2-\bar{\chi}_3}{2}-\frac{1}{2}}.$$

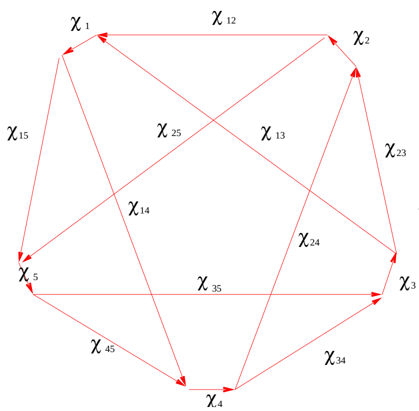
Let $\frac{n_1}{2}$, $\frac{n_2}{2}$ and$\frac{n_3}{2}$ are the real parts of χ_1, χ_2 and χ_2. The above result is to be replaced by zero if the sum $n_1+n_2+n_3$ is not even (please see appendix B). Usually in the $SU(2)$ spin networks the internal link stands for a summation over the azimuthal quantum numbers. Here it is replaced by integration over the z's using the Riemann measure on the complex plane $dz \wedge d\bar{z}$.

Now a quantum amplitude can be calculated for any closed simplicial 3-surface made of tetrahedra in the same way as it is calculated for a closed 2-surface using the Clebsch-Gordan coefficients of $SU(2)$ [18]. The 4-simplex amplitude 'the $\{15\chi\}$ symbol' is calculated by evaluating the spin network associated to its boundary.

This simplex amplitude is precisely analogous to the one calculated in the previous section.

2.3 Asymptotic limit of the BF simplex amplitude

The $\{15\chi\}$ symbol is given by

$\{15\chi\} =$ 

.

In this diagram χ_i is the unitary representation associated to the internal link of each edge, χ_{ij} is the representation associated to the bone which is the intersection of i'th edge and j'th edge. The z_i and z_{ij} are similarly defined. The asymptotic limit is the limit of the $\{15\chi\}$ symbol when the χ's are scaled by a real number λ which is sent to infinity [13], [14].

The simplex amplitude is of the form

$$\{15\chi\} = \int a(z_i, z_{ij}) \exp(-i\phi(z_i, z_{ij}, \chi_i, \chi_{ij})) \textstyle\prod_i dz_i \prod_{i<j} dz_{ij}, \tag{2.7}$$

where $a(z_i, z_{ij})$ are factors in $\{15\chi\}$ that do not depend the χ's in the definition of the Clebsch-Gordan coefficients, $\phi(z_i, z_{ij}, \chi_i, \chi_{ij})$ is i times the natural log of the product of the remaining factors. The $\phi(z_i, z_{ij}, \chi_i, \chi_{ij})$ is a real function which is first order in the χ's and $\ln(z_i - z_{jk})$ and takes the form

$$\sum \chi_l \ln(z_i - z_{jk}) + c.c,$$

where summation is over specific choices of the $\chi_l \ln(z_i - z_{jk})$ terms which can be explicitly calculated from the definition of $\{15\chi\}$. The asymptotic limit is the limiting function of the following when $\lambda \longrightarrow \infty$

$$\{15\chi\}_\lambda = \int a(z_i, z_{ij}) \exp(-i\lambda\phi(z_i, z_{ij}, \chi_i, \chi_{ij})) \textstyle\prod_i dz_i \prod_{i<j} dz_{ij}.$$

The limit can be calculated using the geometric asymptotic formula [19],

$$\int a(y) e^{i\lambda\phi(y)} dy = \left(\frac{2\pi}{k}\right)^{\frac{n}{2}} \sum_{y|d\phi(y)=0} e^{\frac{\pi i sgn H(y)}{4}} \frac{e^{ik\phi(y)} a(y)}{\sqrt{\det|H(y)|}} + O(k^{-n/2-1}),$$

where y is an real n-tuple and $H(y)$ is the Hessian of ϕ at y. This implies the asymptotics are controlled by the extremums of $\phi(y)$. In equation (2.7) n is 30.

Since $\phi(z_i, z_{ij}, \chi_i, \chi_{ij})$ is real in equation (2.7), its extrema are determined just by the equations

$$\frac{\partial \phi}{\partial z_i} = 0, \frac{\partial \phi}{\partial z_{ij}} = 0, \forall i, j.$$

These equations are of the form,

$$\sum \frac{\chi_l}{(z_i - z_{jk})} = 0. \tag{2.8}$$

Further work is need to be done to solve and investigate solutions of these equations which we leave as a open problem. We believe this work can help us understand the semiclassical limit of the related gravity and also the relationship between gravity and topological field theories in four dimensions.

3 The Spin Foam Model of SL(2,$\boldsymbol{C}$) Gravity.

In the previous section a spin foam for the Lorentzian BF theory was derived. To reduce this theory to that of gravity further constraints, called the Barrett-Crane constraints given below have to be imposed at the quantum level on the edges [8], [9], [10], [11], [2]

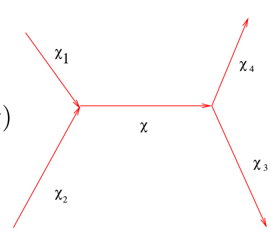
$$B_i \wedge B_j = 0, \forall i, j.$$

The Barrett-Crane constraints are basically the discretized Plebanski constraints [20].

In the BF theory each simplex is an atom of a topology. Imposing the above constraints reduces the atom of a topology to an atom of a geometry. The solution for these equations in the case of Riemannian gravity was proposed by Barrett-Crane, partially proved by Barbieri [21] and completely proved by Reisenberger [11] .

To sovle Barrett-Crane constraints for Lorentzian gravity, consider the following ansatz for the solution of the Barrett-Crane constraints

$$\psi_{\chi_1\chi_2\chi_3\chi_4} = \int d\chi f(\chi)$$



. (3.1)

Now for any $B \in sl(2, \boldsymbol{C})^*$, the quantum operator $\hat{B}$ can be used to construct the two Casimir operators of $SL(2, \boldsymbol{C})$. The mathematics required has been explained by Ruhl in Ref.[22]. The components of $\hat{B}$ are the rotation and the boost generators $\hat{J}_i$ and $\hat{F}_i$ ($i = 1, 2, 3$). The $\hat{\mathrm{B}}$ is given by,

$$\hat{B} = \begin{bmatrix} 0 & F_1 & F_2 & F_3 \\ -F_1 & 0 & J_3 & -J_2 \\ -F_2 & -J_3 & 0 & J_1 \\ -F_3 & J_2 & -J_1 & 0 \end{bmatrix}.$$

Define $F_\pm = F_1 \pm F_2$, $H_\pm = H_1 \pm H_2$. These operators are given explicitly in appendix A. If A_{ab}, B_{cd} are four dimensional antisymmetric tensors define $A \wedge B = \varepsilon^{abcd}\ A_{ab}B_{cd}$. Then the Casimir operators are the following [22]:

$$I_1 = \hat{B} \wedge *\hat{B} = F_+F_- + F_-F_+ + 2F_3^2 - H_+H_- - H_-H_+ - 2H_3^2, \qquad (3.2)$$

$$I_2 = \hat{B} \wedge \hat{B} = H_+F_- + H_-F_+ + F_+H_- + F_-H_+ + 4H_3F_3, \qquad (3.3)$$

where $H_\pm = H_1 \pm iH_2$ and $F_\pm = F_1 \pm iF_2$.

These two operators are proportional to the identity operator in their actions on the functions $\in D_\chi$ with the eigenvalues $\frac{1}{2}\left(\rho^2 - m^2 - 2\right)$ and ρm.

So the classical Barrett-Crane constraint at the quantum level for $i = j$, $\hat{B}_i \wedge \hat{B}_i \psi = 0$, simply states that either ρ_i or m_i is zero. This was first observed by Barrett-Crane [8]. So each i, these constraints can be solved by setting ρ_i or m_i to be zero. The Barrett-Crane model of the Lorentzian quantum gravity corresponds to the case in which $m_i = 0$ for all edges. In the Rovelli-Perez model [5] a specific linear superposition of both of these cases $\rho_i = 0$ and $m_i = 0$, which is sufficient to do harmonic analysis on the single sheet hyperboloid of four dimensional Minkowski space-time [23] was used.

Hence forth we assume that either ρ_i or m_i is zero.

Consider the graph in equation (3.1). The $\hat{B}_1 \wedge \hat{B}_2 \psi = 0$ (and $\hat{B}_3 \wedge \hat{B}_4 \psi = 0$) can be solved as follows. Consider

$$\left(\hat{B}_1 + \hat{B}_2\right) \wedge \left(\hat{B}_1 + \hat{B}_2\right) = \hat{B}_1 \wedge \hat{B}_1 + \hat{B}_2 \wedge \hat{B}_2 + 2\hat{B}_1 \wedge \hat{B}_2.$$

From this it can be inferred that

$$\left(\hat{B}_1 + \hat{B}_2\right) \wedge \left(\hat{B}_1 + \hat{B}_2\right) \psi = 0 \Longrightarrow \hat{B}_1 \wedge \hat{B}_2 \psi = 0.$$

Due to the invariance of the three vertex in equation (3.1), this suggests setting ρ or $m = 0$ where $\chi = m + i\rho$ is the representation of the internal link.

Imposing $\hat{B}_1 \wedge \hat{B}_3 \psi = 0$ (or one of $\hat{B}_1 \wedge \hat{B}_4 \psi = 0$, $\hat{B}_2 \wedge \hat{B}_3 \psi = 0$ and $\hat{B}_2 \wedge \hat{B}_4 \psi = 0$) is a more difficult case. It can be evaluated explicitly in terms of the generators using the identity,

$$\hat{B}_1 \wedge \hat{B}_3 = \frac{\hat{B}_1 \wedge \hat{B}_1 + \hat{B}_3 \wedge \hat{B}_3 - \left(\hat{B}_1 + \hat{B}_3\right) \wedge \left(\hat{B}_1 + \hat{B}_3\right)}{2}. \qquad (3.4)$$

To evaluate the right hand side the results given in equation (3.3) can be used. Then the expression for the generators as operators on the representation space

D_χ derived by Ruhl is substituted. The resulting calculation is cumbersome, but the result is simple and is given by

$$\hat{B}_1 \wedge \hat{B}_3 = (\bar{z}_1 - \bar{z}_3)^2 \frac{\partial}{\partial \bar{z}_1} \frac{\partial}{\partial \bar{z}_3} - (z_1 - z_3)^2 \frac{\partial}{\partial z_1} \frac{\partial}{\partial z_3} + i \left(\frac{m_1 \rho_3 + m_3 \rho_1}{2} - \rho_1 - \rho_3 \right). \tag{3.5}$$

One must solve the equation $\left(\hat{B}_1 \wedge \hat{B}_3\right) \psi = 0$ (and other similar cross constraints) for ψ. This will enable us to construct the general Lorentzian model.

4 Two Issues

4.1 A problem with the representation theory

One of the problems that we encountered in our work is with the expansion of $\delta(g)$. Having a proper expansion for $\delta(g)$ is equivalent to having a proper harmonic analysis for the functions on the group $SL(2, \boldsymbol{C})$. One can derive the following expansion for $\delta(g)$ using the Gelfand-Naĭmark representation theory

$$\delta(g) = \frac{1}{8\pi^4} \int \chi \bar{\chi} tr(\rho_\chi(g)) d\chi.$$

This expansion was first used by Rovelli and Perez [4]. In the right-hand side the $tr(\rho_\chi(g))$ is purely a function of the eigenvalues of g (please see appendix A). Because of this right-hand-side of the above equation is peaked at g's for which the eigen-values are 1. But the eigen-values of g is 1 not only at the identity but also when it is strictly upper or lower triangular. So this means that the right-hand-side does not have the proper expansion of $\delta(g)$. This expansion for $\delta(g)$ has been used here and also in the Lorentzian spin foam derivations by Rovelli and Perez [4], [5]. We believe that this problem has to be fixed or clarified.

4.2 Rigorous derivation of the imposition of the constraints.

In section three we proceeded to derive a spin foam model of Lorentzian quantum gravity by imposing the quantum Barrett-Crane constraints on the BF spin foam model. But both in Riemannian and Lorentzian quantum gravity, this way of imposing of the Barrett-Crane constraints is not yet been rigorously derived using the path integral quantization of any discrete action. There have been various proposals [24], [25] for doing this calculation but they have not yet been fully implemented. Because of this issue we believe the amplitudes of the lower dimensional (< 4) simplices are yet to be fixed (assumming we got the four-simplex amplitudes correct atleast to a certain level). Please read [**?**] for more ideas about this.

5 Conclusion

Here a construction of the general Lorentzian spin foam model has been explained. To finish the model one would have to solve equation (3.5). This will enable us to construct a model which is possibly richer in physical aspects compared to the previously known models. Also we formally derived the equations that describe the asymptotic limit of $SL(2,C)$ BF theory. These equations should be solvable because of their simplicity. Investigating the solutions of these will help us understand the asymptotic limit of the general Lorentzian model. Also they can shed more light on the relation between the topological field theory and gravity in four dimensions. *But before getting oneself into investigating all these, we believe it is important to address the two issues mentioned in the previous section.*

6 Acknowledgements

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A The representation theory of SL(2,$\boldsymbol{C}$)

The Representation theory of $SL(2,\boldsymbol{C})$ was developed by Gelfand and Naĭmark [12], [23]. This was further studied and developed by Ruhl [22]. We recommend [23] to the beginners. The unitary irreducible representations of $SL(2,\boldsymbol{C})$ are infinite dimensional. Each unitary representation of $SL(2,\boldsymbol{C})$ can be described by an appropriate action of the group on functions of single complex variable $\phi(z)$ and is labelled by a complex number $\chi = \frac{n}{2} + i\frac{\rho}{2}$, where n is an integer and ρ is a real number. We denote the linear space on which the representation acts as D_χ. Henceforth we denote χ as a pair (χ_1, χ_2) where $\chi_1 = -\bar{\chi}_2 = \frac{n+i\rho}{2}$. The readers might refer to the references for a fuller descrition of D_χ and all other related mathematical constructs.

Let g is an element of $SL(2,\boldsymbol{C})$ given by

$$g = \begin{bmatrix} \alpha & \beta \\ \gamma & \delta \end{bmatrix},$$

where α,β,γ and δ are complex numbers such that $\alpha\delta - \beta\delta = 1$.

Then the χ representation can be described by action of an unitary operator $T_\chi(g)$ on functions $\phi(z\ )$ of a complex variable z as given below.

$$T_\chi(g)\phi(z) = (\beta z_1 + \delta)^{\chi_1 - 1}(\beta^* z_1^* + \delta^*)^{\chi_2 - 1}\phi(\frac{\alpha z + \gamma}{\beta z + \delta}).$$

This action on $\phi(z)$ is unitary under the inner product defined by

$$(\phi(z), \eta(z)) = \frac{i}{2} \int d^2 z \bar{\phi}(z) \eta(z) d^2 z,$$

where $d^2 z = dz \wedge d\bar{z}$.

The above equation can also be written as,

$$T_\chi(g)\phi(z_1) = \int T_\chi(g)(z_1, z_2)\phi(z_2) d^2 z_2.$$

The $T_\chi(g)(z_1, z_2)$ is defined as

$$T_\chi(g)(z_1, z_2) = (\beta z_1 + \delta)^{\chi - 1}(\beta^* z_1^* + \delta^*)^{-\bar{\chi}-1}\delta(z_2 - g(z_1)), \tag{A.1}$$

where $g(z_1) = \frac{\alpha z_1 + \gamma}{\beta z_1 + \delta}$

The Kernel $T_\chi(g)(z_1, z_2)$ is the analog of the matrix representation of finite dimensional unitary representations of compact groups. We will denote this diagrammatically by $\xrightarrow{\chi} g \longrightarrow$.

An infinitesimal group element a of $SL(2, \boldsymbol{C})$ can be parametrized by six real real numbers ε_k and η_k as shown below [22]

$$a \approx I + \frac{i}{2} \sum_{k=1}^{3} (\varepsilon_k \sigma_k + \eta_k i \sigma_k),$$

where the σ_k are the Pauli matrices. The corresponding six generators of the χ representation are H_k and F_k . The H_k correspond to rotations and the F_k correspond to boosts. These are differential operators on $\phi(z)$ which are defined as

$$F_\pm = F_1 \pm i F_2, \qquad H_\pm = H_1 \pm i H_2,$$

$$H_+ = -z^2 \frac{\partial}{\partial z} - \frac{\partial}{\partial \bar{z}} + (\chi_1 - 1)z, \qquad F_- = i\frac{\partial}{\partial z} - i\bar{z}^2 \frac{\partial}{\partial \bar{z}} + i(\chi_2 - 1)\bar{z},$$

$$H_- = \bar{z}^2 \frac{\partial}{\partial \bar{z}} + \frac{\partial}{\partial z} - (\chi_2 - 1)\bar{z}, \qquad F_+ = -iz^2 \frac{\partial}{\partial z} + i\frac{\partial}{\partial \bar{z}} + i(\chi_1 - 1)z,$$

$$H_3 = z\frac{\partial}{\partial z} - \bar{z}\frac{\partial}{\partial \bar{z}} + \frac{1}{2}(\chi_2 - \chi_1), \qquad F_3 = iz\frac{\partial}{\partial z} + i\bar{z}\frac{\partial}{\partial \bar{z}} - \frac{i}{2}(\chi_2 + \chi_1 - 2).$$

A measure on the group g is given by

$$dg = \left(\frac{i}{2}\right)^3 \frac{d^2\beta d^2\gamma d^2\delta}{|\delta|^2} = \left(\frac{i}{2}\right)^3 \frac{d^2\alpha d^2\beta d^2\gamma}{|\alpha|^2}.$$

This measure is invariant under left translation, right translation and inversion in $SL(2, \boldsymbol{C})$.

The measure can be also written as follows

$$dg = \left(\frac{i}{2}\right)^3 \delta(\alpha\beta - \gamma\delta - 1) d^2\alpha d^2\beta d^2\gamma d^2\delta,$$

where we allow α, β, γ and δ to vary freely while doing integration.

The Fourier transform theory on $SL(2, \boldsymbol{C})$ was developed in [23]. If $f(g)$ is a square integrable function on the group, it has a group Fourier transform defined by

$$F(\chi) = \int f(g) T_\chi(g) dg.$$

The associated inverse Fourier transform is

$$f(g) = \frac{1}{8\pi^4} \int F(\chi) T_\chi(g^{-1}) \chi_1 \chi_2 d\chi.$$

From this it can inferred that

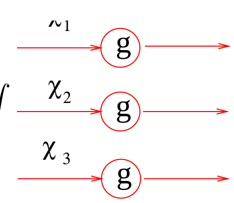
$$\delta(g) = \frac{1}{8\pi^4} \int tr\left[T_\chi(g)\right] \chi_1 \chi_2 d\chi.$$

The above equation was first used by Rovelli and Perez [4], [5] to derive the spin foam models of Lorentzian gravity from the formalism of field theory over group.

B Derivation of the 3D edge integral.

We need to evaluate the following edge integral which is needed to determine the four dimensional edge integral and also the Clebsch-Gordan Coefficients of $SL(2, \boldsymbol{C})$.

$I_3 = \int$



dg

$$= \int T_\chi(g)(x_1, x_2) T_\lambda(g)(y_1, y_2) T_\mu(g)(z_1, z_2) dg$$

$$= \int (\beta x_1 + \delta)^{\chi_1 - 1} (\beta^* x_1^* + \delta^*)^{\chi_2 - 1} \delta(x_2 - \frac{\gamma + x_1\alpha}{\delta + x_1\beta})$$

$$(\beta y_1 + \delta)^{\lambda_1 - 1} (\beta^* y_1^* + \delta^*)^{\lambda_2 - 1} \delta(y_2 - \frac{\gamma + y_1\alpha}{\delta + y_1\beta})$$

$$(\beta z_1 + \delta)^{\mu_1 - 1} (\beta^* z_1^* + \delta^*)^{\mu_2 - 1} \delta(z_2 - \frac{\gamma + z_1\alpha}{\delta + z_1\beta}) dg$$

$$
\begin{aligned}
&= \int (\beta x_1 + \delta)^{\chi_1} (\beta^* x_1^* + \delta^*)^{\chi_2} \delta((\delta + x_1\beta)\, x_2 - \gamma - x_1\alpha) \\
&(\beta z_1 + \delta)^{\mu_1} (\beta^* z_1^* + \delta^*)^{\mu_2} \delta((\delta + z_1\beta)\, z_2 - \gamma - z_1\alpha) \\
&(\beta y_1 + \delta)^{\lambda_1} (\beta^* y_1^* + \delta^*)^{\lambda_2} \delta((\delta + y_1\beta)\, y_2 - \gamma - y_1\alpha) \delta\, (\gamma\beta - \alpha\delta + 1) \\
&d^2\alpha d^2\beta d^2\gamma d^2\delta.
\end{aligned}
$$

Let us define the variables $a = \frac{\alpha}{\gamma}, b = \frac{\beta}{\gamma}$ and $c = \frac{\delta}{\gamma}$.

To eliminate the deltas we need to solve the following linear equations for a, b and c :

$$
\begin{aligned}
(c + x_1 b)\, x_2 - x_1 a = 1, \\
(c + y_1 b)\, y_2 - y_1 a = 1, \\
(c + z_1 b)\, z_2 - z_1 a = 1.
\end{aligned}
$$

The determinant of the above system is

$$
D = x_1 x_2 y_1 z_2 - x_1 x_2 y_2 z_1 - x_1 y_1 y_2 z_2 + x_2 y_1 y_2 z_1 + x_1 y_2 z_1 z_2 - x_2 y_1 z_1 z_2.
$$

The solution of the linear system is given by

$$
\begin{aligned}
a &= \frac{x_1 x_2 y_2 - x_1 x_2 z_2 - x_2 y_1 y_2 + x_2 z_1 z_2 + y_1 y_2 z_2 - y_2 z_1 z_2}{x_1 x_2 y_1 z_2 - x_1 x_2 y_2 z_1 - x_1 y_1 y_2 z_2 + x_2 y_1 y_2 z_1 + x_1 y_2 z_1 z_2 - x_2 y_1 z_1 z_2}, \\
b &= \frac{x_2 y_1 - x_1 y_2 + x_1 z_2 - x_2 z_1 - y_1 z_2 + y_2 z_1}{x_1 x_2 y_2 z_1 - x_1 x_2 y_1 z_2 + x_1 y_1 y_2 z_2 - x_2 y_1 y_2 z_1 - x_1 y_2 z_1 z_2 + x_2 y_1 z_1 z_2}, \\
c &= \frac{x_1 x_2 z_1 - x_1 x_2 y_1 + x_1 y_1 y_2 - x_1 z_1 z_2 - y_1 y_2 z_1 + y_1 z_1 z_2}{x_1 x_2 y_2 z_1 - x_1 x_2 y_1 z_2 + x_1 y_1 y_2 z_2 - x_2 y_1 y_2 z_1 - x_1 y_2 z_1 z_2 + x_2 y_1 z_1 z_2}.
\end{aligned}
$$

Let us define the variables,

$$
\begin{aligned}
A &= (x_1 - z_1)\,(y_1 - x_1)\,(z_2 - y_2)\,, \\
B &= (x_2 - z_2)\,(z_1 - y_1)\,(y_1 - x_1)\,, \\
C &= (y_2 - x_2)\,(y_1 - z_1)\,(z_1 - x_1)\,, \\
t &= \frac{\gamma}{D}, \\
E &= (z_2 - x_2)\,(z_2 - y_2)\,(x_2 - y_2)\,(z_1 - y_1)\,(z_1 - x_1)\,(y_1 - x_1)\,.
\end{aligned}
$$

Then using the expressions for a, b and c we can derive the following results,

$$
\begin{aligned}
c + x_1 b &= D^{-1} A, \\
c + y_1 b &= D^{-1} B, \\
c + z_1 b &= D^{-1} C, \\
b - ac &= D^{-2} E.
\end{aligned}
$$

Using the above results and definitions, I can simplified as,

$$I_3 = A^{\chi_1}\bar{A}^{\chi_2}B^{\lambda_1}\bar{B}^{\lambda_2}C^{\mu_1}\bar{C}^{\mu_2}\int \delta\left(t^2E+1\right)t^{\chi_1+\lambda_1+\mu_1}\bar{t}^{\chi_1+\lambda_1+\mu_1}d^2t.$$

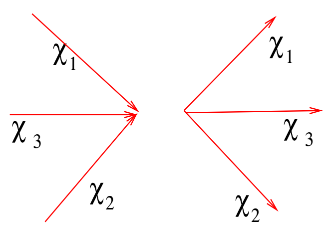
Let $\frac{n_1}{2}$,$\frac{n_2}{2}$ and$\frac{n_3}{2}$ be the imaginary parts of χ_1, λ_1 and μ_1. We can solve for t from $t^2E+1=0$ as $\pm E^{-\frac{1}{2}}$. Substituting this we find I_3 is not zero only if $n_1+n_2+n_3$ is even. Hence forth let us assume that $n_1+n_2+n_3$ is even.

After the final integration we get

$$I_3 = 2A^{\chi_1}\bar{A}^{\chi_2}B^{\lambda_1}\bar{B}^{\lambda_2}C^{\mu_1}\bar{C}^{\mu_2}\left|E\right|^{-1}E^{-\frac{\chi_1+\lambda_1+\mu_1}{2}}\bar{E}^{-\frac{\chi_1+\lambda_1+\mu_1}{2}}.$$

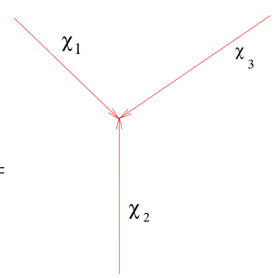
From this I_3 can been explicitly expressed as

$$\begin{aligned} I_3 &= C(\mu_1,\chi_1,\lambda_1,x_1,y_1,z_1)C(-\mu_2,-\chi_2,-\lambda_2,x_2,y_2,z_2) \\ &= C(\mu_1,\chi_1,\lambda_1,x_1,y_1,z_1)\bar{C}(\mu_2,\chi_2,\lambda_2,x_2,y_2,z_2) \end{aligned}$$

$=$ 

,

where C is the Clebsch-Gordan coefficient defined by

$C(\mu,\chi,\lambda,x,y,z) =$ 

$$= (x-y)^{\frac{-\chi-\lambda+\mu}{2}-\frac{1}{2}}(z-x)^{\frac{-\chi+\lambda-\mu}{2}-\frac{1}{2}}(y-z)^{\frac{\chi-\lambda-\mu}{2}-\frac{1}{2}}$$
$$\overline{(x-y)}^{-\frac{-\bar{\chi}-\bar{\lambda}+\bar{\mu}}{2}-\frac{1}{2}}\overline{(z-x)}^{-\frac{-\bar{\chi}+\bar{\lambda}-\bar{\mu}}{2}-\frac{1}{2}}\overline{(y-z)}^{-\frac{\bar{\chi}-\bar{\lambda}-\bar{\mu}}{2}-\frac{1}{2}}.$$

We recite again that last result is to be replaced by zero if $n_1+n_2+n_3$ is not even.

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